Essay

Thoughts on Modification of Bio-harmony

Matti Pitkänen [1]

Abstract

I have constructed a model of bio-harmony as a fusion of 3 icosahedral harmonies and tetrahedral harmony. The icosahedral harmonies are defined by Hamiltonian cycles at icosahedron going through every vertex of the icosahedron and therefore assigning to each triangular face an allowed 3-chord of the harmony. The fascinating outcome is that the model can reproduces genetic code. The model for how one can understand how 12-note scale can represent 64 genetic codons has the basic property that each note belongs to 16 chords. The reason is that there are 3 disjoint sets of notes and given 3-chord is obtained by taking 1 note from each set. For bio-harmony obtained as union of 3 icosahedral harmonies and tetrahedral harmony note typically belongs to 15 chords. The representation in terms of frequencies however requires 16 chords per note. Consistency a modification of the model of icosahedral harmony. The necessity to introduce tetrahedron for one of the 3 fused harmonies is indeed an ugly looking feature of the model. The question is whether one of the harmonies could be replaced with some other harmony with 12 notes and 24 chords. If this would work one would have 64 chords equal to the number of genetic codons and 5+5+6 =16 chords per note. One can imagine toric variants of harmonies realized in terms of Hamiltonian cycles and one indeed obtains a toric harmony with 12 notes and 24 3-chords. Bio-harmony could correspond to the fusion of 2 icosahedral harmonies with 20 chords and toric harmony with 24 chords having therefore 64 chords. Whether the predictions for the numbers of codons coding for given amino-acids come out correctly for some choices of Hamiltonian cycles is still unclear.

Keywords: Bio-harmony, modification, genetic code, TGD framework.

1 Introduction

I have developed a rather detailed model of bio-harmony as a fusion of 3 icosahedral harmonies and tetrahedral harmony [3, 4](see `http://tinyurl.com/yad4tqwl` and `http://tinyurl.com/y8njuctq`). The icosahedral harmonies are defined by Hamiltonian cycles at icosahedron going through every vertex of the icosahedron and therefore assigning to each triangular face an allowed 3-chord of the harmony. The surprising outcome is that the model can reproduces genetic code.

The model for how one can understand how 12-note scale can represent 64 genetic codons has the basic property that each note belongs to 16 chords. The reason is that there are 3 disjoint sets of notes and given 3-chord is obtained by taking 1 note from each set. For bio-harmony obtained as union of 3 icosahedral harmonies and tetrahedral harmony note typically belongs to 15 chords. The representation in terms of frequencies requires 16 chords per note.

If one wants consistency one must somehow modify the model of icosahedral harmony. The necessity to introduce tetrahedron for one of the 3 fused harmonies is indeed an ugly looking feature of the model. The question is whether one of the harmonies could be replaced with some other harmony with 12 notes and 24 chords. If this would work one would have 64 chords equal to the number of genetic codons and 5+5+6 =16 chords per note. The addition of tetrahedron would not be needed.

One can imagine toric variants of harmonies realized in terms of Hamiltonian cycles and one indeed obtains a toric harmony with 12 notes and 24 3-chords. Bio-harmony could correspond to the fusion of 2 icosahedral harmonies with 20 chords and toric harmony with 24 chords having therefore 64 chords. Whether the predictions for the numbers of codons coding for given amino-acids come out correctly for

[1] Correspondence: Matti Pitkänen `http://tgdtheory.fi/`. Address: Rinnekatu 2-4 8A, 03620, Karkkila, Finland. Email: matpitka6@gmail.com.

some choices of Hamiltonian cycles is still unclear. This would require an explicit construction of toric Hamiltonian cycles.

Before discussing the possible role of toric harmonies some previous results will be summarized.

1.1 Icosahedral bio-harmonies

The model of bio-harmony [3] starts from a model for music harmony as a Hamiltonian cycle at icosahedron having 12 vertices identified as 12 notes and 20 triangular faces defining the allowed chords of the harmony. The identification is determined by a Hamiltonian cycle going once through each vertex of icosahedron and consisting of edges of the icosahedral tesselation of sphere (analog of lattice): each edge corresponds to quint that is scaling of the frequency of the note by factor $3/2$ (or by factor $2^{7/12}$ in well-tempered scale). This identification assigns to each triangle of the icosahedron a 3-chord. The 20 faces of icosahedron define therefore the allowed 3-chords of the harmony. There exists quite a large number of icosahedral Hamiltonian cycles and thus harmonies.

The fact that the number of chords is 20 - the number of amino-acids - leads to the question whether one might somehow understand genetic code and 64 DNA codons in this framework. By combining 3 icosahedral harmonies with different symmetry groups identified as subgroups of the icosahedral group, one obtains harmonies with 60 3-chords.

The DNA codons coding for given amino-acid are identified as triangles (3-chords) at the orbit of triangle representing the amino-acid under the symmetry group of the Hamiltonian cycle. The predictions for the numbers of DNAs coding given amino-acid are highly suggestive for the vertebrate genetic code.

By gluing to the icosahedron tetrahedron along common face one obtains 4 more codons and two slightly different codes are the outcome. Also the 2 amino-acids Pyl and Sec can be understood. One can also regard the tetrahedral 4 chord harmony as additional harmony so that one would have fusion of four harmonies. One can of course criticize the addition of tetrahedron as a dirty trick to get genetic code.

The explicit study of the chords of bio-harmony however shows that the chords do not contain the 3-chords of the standard harmonies familiar from classical music (say major and minor scale and corresponding chords). Garage band experimentation with random sequences of chords requiring conservability that two subsequent chords have at least one common note however shows that these harmonies are - at least to my opinion - aesthetically feasible although somewhat boring.

1.2 Explanation for the number 12 of notes of 12-note scale

One also ends up to an argument explaining the number 12 for the notes of the 12-note scale [3]. There is also second representation of genetic code provided by dark proton triplets. The dark proton triplets representing dark genetic codons are in one-one correspondence with ordinary DNA codons. Also amino-acids, RNA and tRNA have analogs as states of 3 dark protons. The number of tRNAs is predicted to be 40.

The dark codons represent entangled states of protons and one cannot decompose them into a product state. The only manner to assign to the 3-chord representing the triplet ordinary DNA codon such that each letter in {A,T,C,G} corresponds to a frequency is to assume that the frequency depends on the position of the letter in the codon. One has altogether $3 \times 4 = 12$ frequencies corresponding to 3 positions for given letter selected from four letters.

Without additional conditions any decomposition of 12 notes of the scale to 3 disjoint groups of 4 notes is possible and possible chords are obtained by choosing one note from each group. The most symmetric choice assigns to the 4 letters the notes $\{C, C\sharp, D, D\sharp\}$ in the first position, $\{E, F, F\sharp, G\}$ in the second position, and $\{G\sharp, A, B\flat, B\}$ in the third position. The codons of type XXX would correspond to $CEG\sharp$ or its transpose. One can transpose this proposal and there are 4 non-quivalent transposes, which could be seen as analogs of music keys.

Remark: $CEG\sharp$ between C-major and A-minor very often finishes finnish tango: something neither sad nor glad!

One can look what kind of chords one obtains.

1. Chords containing notes associated with the same position in codon are not possible.

2. Given note belongs to 6 chords. In the icosahedral harmony with 20 chords given note belongs to 5 chords (there are 5 triangles containing given vertex). Therefore the harmony in question cannot be equivalent with 20-chord icosahedral harmony. Neither can the bio-harmony with 64 chords satisfy the condition that given note is contained by 6 3-chords.

3. First and second notes of the chords are separated by at least major third as also those second and third notes. The chords satisfy however octave equivalence so that the distance between the first and third notes can be smaller - even half step - and one finds that one can get the basic chords A-minor scale: Am, Dm, E7, and also G and F. Also the basic chords of F-major scale can be represented. Also the transposes of these scales by 2 whole steps can be represented so that one obtains A_m, $C\sharp_m$, F_m and corresponding major scales. These harmonies could allow the harmonies of classical and popular music.

These observations encourage to ask whether a representation of the new harmonies as Hamiltonian cycles of some tesselation could exist. The tesselation should be such that 6 triangles meet at given vertex. Triangular tesselation of torus having interpretation in terms of a planar parallelogram (or perhaps more general planar region) with edges at the boundary suitable identified to obtain torus topology seems to be the natural option. Clearly this region would correspond to a planar lattice with periodic boundary conditions.

2 Is it possible to have toric harmonies?

The basic question is whether one can have a representation of the new candidate for harmonies in terms of a tesselation of torus having $V = 12$ vertices and $F = 20$ triangular faces. The reading of the article "*Equivelar maps on the torus*" [1] (see `http://tinyurl.com/ya6g9kwe`) discussing toric tesselations makes clear that this is impossible. One however have $(V, F) = (12, 24)$ (see `http://tinyurl.com/y7xfromc`). A rather promising realization of the genetic code in terms of bio-harmony would be as a fusion of two icosahedral harmonies and toric harmony with $(V, F) = (12, 24)$. This in principle allows also to have 24 3-chords which can realize classical harmony (major/minor scale).

1. The local properties of the tesselations for any topology are characterized by a pair (m, n) of positive integers. m is the number of edges meeting in given vertex (valence) and n is the number of edges and vertices for the face. Now one has $(m, n) = (6, 3)$. The dual of this tesselation is hexagonal tesselation $(m, n) = (3, 6)$ obtained by defining vertices as centers of the triangles so that faces become vertices and vice versa.

2. The rule $V - E + F = 2(1 - g) - h$, where V, E and F are the numbers of vertices, edges, and faces, relates $V - E - F$ to the topology of the graph, which in the recent case is triangular tesselation. g is the genus of the surface at which the triangulation is im eded and h is the number of holes in it. In case of torus one would have $E = V + F$ giving in the recent case $E = 36$ for $(V, F) = (12, 24)$ (see `http://tinyurl.com/y7xfromc`) whereas in the icosahedral case one has $E = 32$.

3. This kind of tesselations are obtained by applying periodic boundary conditions to triangular lattices in plane defining parallelogram. The intuitive expectation is that this lattices can be labelled by two integers (m, n) characterizing the lengths of the sides of the parallelogram plus angle between two sides: this angle defines the conformal equivalence class of torus. One can also introduce two unit vectors e_1 and e_2 characterizing the conformal equivalence class of torus.

Second naive expectation is that $m \times n \times sin(\theta)$ represents the area of the parallelogram. $sin(\theta)$ equals to the length of the exterior product $|e_1 \times e_2| = sin(\theta)$ representing twice the area of the triangle so that there would be $2m \times n$ triangular faces. The division of the planar lattice by group generated by $pe_1 + qe_2$ defines boundary conditions. Besides this the rotation group Z_6 acts as analog for the symmetries of a unit cell in lattice. This naive expectation need not of course be strictly correct.

4. As noticed, it is not possible to have triangular toric tesselations with $(V, E, F) = (12, 30, 20)$. Torus however has a triangular tesselation with $(V, E, F) = (12, 36, 24)$. An illustration of the tesselation can be found at `http://tinyurl.com/y7xfromc`. It allows to count visually the numbers V, E, F, and the identifications of the boundary edges and vertices. With good visual imagination one might even try to guess what Hamiltonian cycles look like.

   The triangular tesselations and their hexagonal duals are characterized partially by a pair of integers (a, b) and (b, a). a and b must both even or odd (see `http://tinyurl.com/y7xfromc`). The number of faces is $F = (a^2 + 3b^2)/2$. For $(a, b) = (6, 2)$ one indeed has $V = 12$ and $F = 24$. From the article [1] (see `http://tinyurl.com/ya6g9kwe`) one learns that the number of triangles satisfies $F = 2V$ for $p = q$ at least. If $F = 2V$ holds true more generally one has $V = (a^2 + 3b^2)/8$, giving a tight constraints on a and b.

   Remark: The conventions for the labelling of torus tesselation vary. The above convention based on integers (a, b) used in the illustrations at `http://tinyurl.com/y7xfromc` is different from the convention based on integer pair (p, q) used in [1] . In this notation torus tesselation with $(V, F) = (12, 24)$ corresponds to $(p, q) = (2, 2)$ instead of $(a, b) = (6, 2)$. This requires $(a, b) = (3p, q)$. With these conventions one has $V = p^2 + q^2 + pq$.

2.1 The number of triangles in the 12-vertex tesselation is 24: curse or blessing?

One could see as a problem that one has $F = 24 > 20$? Or is this a problem?

1. By fusing two icosahedral harmonies and one toric harmony one would obtain a harmony with 20+20+24 =64 chords, the number of DNA codons! One would replace the fusion of 3 icosahedral harmonies and tetrahedral harmony with a fusion of 2 icosahedral harmonies and toric harmony. Icosahedral symmetry with toric symmetry associated with the third harmony would be replaced with a smaller toric symmetry. Note however that the attachment of tetrahedron to a fixed icosahedral face also breaks icosahedral symmetry.

   This raises questions. Could the presence of the toric harmony somehow relate to the almost exact $U \leftrightarrow C$ and $A \leftrightarrow G$ symmetries of the third letter of codons. This does not of course mean that one could associated the toric harmony with the third letter. Note that in the icosa-tetrahedral model the three harmonies are assumed to have no common chords. Same non-trivial assumption is needed also now in order to obtain 64 codons.

2. What about the number of amino-acids: could it be 24 corresponding ordinary aminoacids, stopping sign plus 3 additional exotic amino-acids. The 20 icosahedral triangles can corresponds to amino-acids but not to stopping sign. Could it be that one of the additional codons in 24 corresponds to stopping sign and two exotic amino-acids Pyl and Sec appearing in biosystems explained by the icosahedral model in terms of a variant of the genetic code. There indeed exists even third exotic amino-acid! N-formylmethionine (see `http://tinyurl.com/jsphvgt`) but is usually regarded as as a form of methionine rather than as a separate proteinogenic amino-acid.

3. Recall that the problem related to the icosa-tetrahedral harmony is that it does not contains the chords of what might be called classical harmonies (the chordds assignable to major and minor

scales). If 24 chords of bio-harmony correspond to toric harmony, one could obtain these chords if the chords in question are chords obtainable by the proposed construction.

But is this construction consistent with the representation of 64 chords by taking to each chord one note from 3 disjoint groups of 4 notes in which each note belongs to 16 chords. The maximum number of chords that note can belong to would be 5+5+6=16 as desired. If there are no common chords between the 3 harmonies the conditions is satisfied. Using for instance 3 toric representations the number would be 6+6+6=18 and would require dropping some chords.

4. The earlier model for tRNA as fusion of two icosahedral codes predicting 20+20=40 tRNA codons. Now tRNAs as fusion of two harmonies allows two basic options depending on whether both harmonies are icosahedral or whether second harmony is toric. These options would give 20+20=40 or 20+24=44 tRNAs. Wikipedia tells that maximum number is 41. Some sources however tell that there are 20-40 different tRNAs in bacterial cells and as many as 50-100 in plant and animal cells.

2.2 A more detailed model for toric harmonies

One can consider also more detailed model for toric harmonies.

1. The above discussed representation in terms of frequencies assigned with nucleotides depending on their position requires the decomposition of the notes to 3 disjoint groups of 4 notes. This means decomposition of 12 vertices of Hamiltonian cycle to 4 disjoint groups such that within given group the distances between the members of group are larger than one unit so that they cannot belong to same triangle. There are $Bin(12,4) \times Bin(8,4)$ decomposition to 3 disjoint groups of for vertices, where $Bn(n,k) = n!/(n-k)!k!$ is binomial coefficient.

2. Once the Hamiltonian cycle has been fixed and is one assumes that single step along cycle corresponds to quint, one knows what the notes associated with each vertex is and given the note of the 12-note scale one knows the number$0 \leq n < 12$ of quint steps needed to obtain it. For instance, for the proposed grouping $\{C, C\sharp, D, D\sharp\}$ and its two transposes by 2 hole steps one can assign 4 integers to each group. The condition is that within each group the notes labelled by the integers have minimum distance of 2 units between themselves.

3. One could try to understand the situation in terms of the symmetries of the system.
   (a) Could the triplet $\{C, E, G\sharp\}$ and its four translates be interpreted as Z_3 orbits. Could suitable chosen members from 4 disjoint quartets quite general form Z_3 orbits.
   Remark: Particle physicists notes the analogy with 4 color triplets formed by u and d quarks having spin 1/2. Z_4 would correspond to spin and color spin and Z_3 to color.
   (b) Z_4 acts as symmetries of the tesselation considered and these symmetries respect distances so that their action on a quartet with members having mutual distances larger than unit creates new such quartet. Could the triplet $\{C, E, G\sharp\}$ and its four translates by an $n-$multiple of half note, $n = 0, 1, 2, 3$ correspond to an orbit Z_4?
   Could the groups of 4 notes quite generally correspond to the orbits of Z_4? This can be true only if the action of non-trivial Z_4 elements relates only vertices with distance larger than one unit.

4. The group of isometries of the toric triangulation acts as symmetries. $Z_{24} = Z_6 \times Z_4$ is a good candidate for this group. Z_6 corresponds to the rotations of around given point of triangulation and should leave the tesselation invariant. The orbit of given triangle defining the set of DNA codons coding the amino-acid represented by the orbit would correspond to orbit of subgroups of Z_{24}. Only orbits containing orbits containing $1, 2, 3, 4$ or 6 triangles are allowed by the degeneracies of the genetic code. These numbers would correspond to degeneracies that is the numbers of codons coding for given amino-acid. All these numbers appear as degeneracies.

2.3 What one can say about toric Hamiltonian cycles?

First some basic notions are in order. The graph is said to be equivelar if it is a triangulation of a surface meaning that it has 6 edges emanating from each vertex and each face has 3 vertices and 3 edges [1]. Equivelarity is equivalent with the folllowing conditions;

1. Every vertex is 6-valent.
2. The edge graph is 6-connected.
3. The graph has vertex transitive automorphism group.
4. The graph can be obtained as a quotient of the universal covering tesselation (3,6) by a sublattice (subgroup of translation group). 6-connectedness means that one can decompose the tesselation into two disconnected pieces by removing 6 or more vertices
5. Edge graph is n-connected if the elimation of $k < n$ vertices leaves it connected. It is known that every 5-connected triangulation of torus is Hamiltonian [2] (see `http://tinyurl.com/y7cartk2`). Therefore also 6-connected $(6,3)_{p=2,q=2}$ tesselation has Hamiltonian cycles.
6. The Hamiltonian cycles for the dual tesselation are not in any sense duals of those for the tesselation. For instance, in the case of dodecahedron there is unique Hamiltonian cycle and for icosahedron has large number of cycles. Also in the case of $(6,3)$ tesselations the duals have different Hamilton cycles. In fact, the problem of constructing the Hamiltonian cycles is NP complete.

Can one say anything about the number of Hamiltonian cycles?

1. For dodecahedron only 3 edges emanates from a given vertex and there is only one Hamiltonian cycle. For icosahedron 5 edges emanate from given vertex and the number of cycles is rather large. Hence the valence and also closely related notion of n-connectedness are essential for the existence of Hamilton's cycles. For instance, for a graph consisting of two connected graphs connected by single edge, there exist no Hamilton's cycles. For toric triangulations one has as many as 6 edges from given vertex and this favors the formation of a large number of Hamiltonian cycles.
2. Curves on torus are labelled by winding numbers (M,N) telling the homology equivalence class of the cycle. M and M can be any integers. Curve winds M (N) times around the circle defining the first (second) equivalence homology equivalence class. Also Hamiltonian cycles are characterized by their homology equivalence class, that is pair (M,N) of integers. Since there are only $V = 12$ points, the numbers (M,N) are finite. By periodic boundary conditions means that the translations by multiples of $2e_1 + 2e_2$ do not affect the tesselation (one can see what this means geometrically from the illustration at `http://tinyurl.com/y7xfromc`). Does this mean that (M,N) belongs to $Z_2 \times Z_2$ so that one would have 4 homologically non-equivalent paths.

   Are all four homology classes realized as Hamiltonian cycles? Does given homology class contain several representatives or only single one in which case one would have 20 non-equivalent Hamiltonian cycles?

It turned out that there exist programs coding for an algorithm for finding whether given graph (much more general than tesselation) has Hamiltonian cycles. Having told to Jebin Larosh about the problem, he sent within five minutes a link to a Java algorithm allowing to show whether a given graph is Hamiltonian (see `http://tinyurl.com/y7y9tr5t`): sincere thanks to Jebin! By a suitable modification this algorithm find all Hamiltonian cycles.

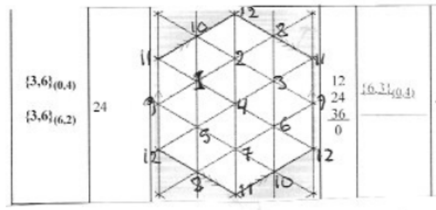

Figure 1: The number of the vertices of $(V, F) = 12, 24)$ torus tesselation allowing path $(0, 1, 2, 3, 4, 6, 5, 8, 10, 7, 11, 9, 0)$ as one particular Hamiltonian cycle.

1. The number N_H of Hamiltonian cycles is expected to be rather large for a torus triangulation with 12 vertices and 24 triangles and it is indeed so: $N_H = 27816$! The image of the tessellation and the numbering of its vertices are described in figure below (see **Fig. 1**). Incide matrix A characterizes the graph: if vertices i and j are connected by edge, one has $A_{ij} = A_{ji} = 1$, otherwise $A_{ij} = Aji = 0$ and is used as data in the algorithm finding the Hamiltonian cycles.

   The cycles related by the isometries of torus tessellation are however equivalent. The guess is that the group of isometries is $G = Z_{2,refl} \rtimes (Z_{4,tr} \rtimes Z_{n,rot})$. $Z_{n,rot}$ is a subgroup of local $Z_{6,rot}$. A priori $n \in \{1, 2, 3, 6\}$ is allowed.

   On basis of [1] I have understood that one has $n = 3$ but that one can express the local action of $Z_{6,rot}$ as the action of the semidirect product $Z_{2,refl} \times Z_{3,rot}$ at a point of tesselation (see `http://tinyurl.com/ya6g9kwe`). The identity of the global actions $Z_{2,refl} \times Z_{3,rot}$ and $Z_{6,rot}$ does not look feasible to me. Therefore $G = Z_{2,refl} \rtimes (Z_{4,tr} \rtimes Z_{3,rot})$ with order $ord(G) = 24$ will be assumed in the following (note that for icosahedral tesselation one has $ord(G) = 120$ so that there is symmetry breaking).

   Z_4 would have as generators the translations e_1 and e_2 defining the conformal equivalence class of torus. The multiples of $2(e_1 + e_2)$ would leave the tesselation invariant. If these arguments are correct, the number of isometry equivalence classes of cycles would satisfy $N_{H,I} \geq N_H/24 = 1159$.

2. The actual number is obtained as sum of cycles characterized by groups $H \subset Z_{12}$ leaving the cycle invariant and one can write $N_{H,I} = \sum_H (ord(H)/ord(G))N_0(H)$, where $N_0(H)$ is the number of cycles invariant under H.

What can one say about the symmetry group H for the cycle?

1. Suppose that the isometry group G leaving the tesselation invariant decomposes into semi-direct product $G = Z_{2,refl} \rtimes (Z_{4,tr} \rtimes Z_{3,rot})$, where $Z_{3,rot}$ leaves invariant the starting point of the cycle. The group H decomposes into a semi-direct product $H = Z_{2,refl} \rtimes (Z_{m,tr} \times Z_{3,rot})$ as subgroup of $G = Z_{2,refl} \rtimes (Z_{4,tr} \times Z_{3,rot})$.

2. $Z_{n,rot}$ associated with the starting point of cycle must leave the cycle invariant at each point. Applied to the starting point, the action of H, if non-trivial - that is $Z_{3,rot}$, must transform the outgoing edge to incoming edge. This is not possible since Z_3 has no idempotent elements so that one can have only $n = 1$. This gives $H = Z_{2,refl} \rtimes (Z_{m,tr}$. $m = 1, 2$ and $m = 4$ are possible.

3. Should one require that the action of H leaves invariant the starting point defining the scale associated with the harmony? If this is the case, then only the group $H = Z_{2,refl}$ would remain and invariance under Z_{refl} would mean invariance under reflection with respect to the axis defined by e_1

or e_2. The orbit of triangle under $Z_{2,refl}$ would consist of 2 triangles always and one would obtain 12 codon doublets instead of 10 as in the case of icosahedral code.

If this argument is correct, the possible symmetry groups H would be Z_0 and $Z_{2,refl}$. For icosahedral code both Z_{rot} and Z_{2_refl} occur but $Z_{2,refl}$ does not occur as a non-trivial factor of H in this case.

The almost exact $U \leftrightarrow C$ and $A \leftrightarrow G$ symmetry of the genetic code would naturally correspond to $Z_{2,refl}$ symmetry. Therefore the predictions need not change from those of the icosahedral model except that the 4 additional codons emerge more naturally. The predictions would be also essentially unique.

4. If H is trivial Z_1, the cycle would have no symmetries and the orbits of triangles would contain only one triangle and the correspondence between DNA codons and amino-acids would be one-to-one. One would speak of disharmony. Icosahedral Hamiltonian cycles can also be of this kind. If they are realized in the genetic code, the almost exact $U \leftrightarrow C$ and $A \leftrightarrow G$ symmetry is lost and the degeneracies of codons assignable to 20+20 icosahedral codons increase by one unit so that one obtains for instance degeneracy 7 instead of 6 not realized in Nature.

What can one say about the character of toric harmonies on basis of this picture.

1. It has been already found that the proposal involving three disjoint quartets of subsequent notes can reproduce the basic chords of basic major and minor harmonies. The challenge is to prove that it can be assigned to some Hamiltonian cycle(s). The proposal is that the quartets are obtained by Z_{rot}^3 symmetry from each other and that the notes of each quartet are obtained by $Z_{4,tr}$ symmetry.

2. A key observation is that classical harmonies involve chords containing 1 quint but not 2 or no quints at all. The number of chords in torus harmonies is $24 = 2 \times 12$ and twice the number of notes. The number of intervals in turn is 36, 3 times the number of the notes. This allows a situation in which each triangle contains one edge of the Hamiltonian cycle so that all 3-chords indeed have exactly one quint.

3. By the above argument harmony possesses Z_2 symmetry or no symmetry at all and one has 12 codon doublets. For these harmonies each edge of cycle is shared by two neighboring triangles containing the same quint. A possible identification is as major and minor chords with same quint. The changing of the direction of the scale and the reflection with respect to the edges the Hamiltonian cycle would transforms major chords and minor chords along it to each other and change the mood from glad to sad and vice versa.

   The proposed harmony indeed contains classical chords with one quint per chord and for $F, A, C\sharp$ both minor and major chords are possible. There are 4 transposes of this harmony.

4. Also Hamiltonian cycles for which n triangles contain two edges of Hamiltonian path (CGD type chords) and n triangles contain no edges. This situation is less symmetric and could correspond to a situtation without any symmetry at all.

5. One can ask whether the classical harmonies corresponds to 24 codons assignable to the toric harmony and to the 24 amino-acids being thus realizable using only amino-acids. If so, the two icosahedral harmonies would represent kind of non-classical exotics.

3 Appendix: Some facts about toric tesselations

Genus $g = 1$ (torus) is unique in that it allows infinite number of tesselations as analogs of planar lattices with periodicic boundary conditions. $g = 0$ allows only Platonic solids as tesselations and $g > 1$ allows very few tesselations. The article [1] gives a nice review about toric tesselations.

1. Toric tesselations correspond to tesselations of plane by periodic boundary conditions. Torus tesselation allows a universal covering identifiable as counterpart of infinite lattice in plane. There are infinite number of coverings of given tesselation labelled by two integers (m, n) since the homology group of torus is $Z \times Z$. The tesselation is obtained by dividing $Z \times Z$ by its normal subgroup. Also the rotation group Z_6 acts as group leaving the tesselation invariant and correspond to the rotation leaving invariant the lattice cell consisting of 6 vertices around given vertex.

2. The tesselation is called decomposable if there is a k-sheeted covering map (map corresponds to a collection of charts) characterized by the subgroup of the isometries of the covering of the tesselation which corresponds to a sub-tesselation. This subgroup is charactrized by a pair (p, q) of integers being generated by the translation $pe_1 + qe_2$ and $2\pi/6$ rotation. The unit vectors can be chosen to be $e_1 = (1, 0)$ and $e_2 = (1, \sqrt{3})/2$ for triangular tesselation (presumably this tesselation is regular tesselation with the conformal equivalence class of torus fixed by the angle between e_1 and e_2). Line reflection transforms $(3, 6)_{p,q}$ to $(3, 6)_{q,p}$ (see Fig 1 of `http://tinyurl.com/ya6g9kwe`). The tesselation is invariant under reflections - regular -if $pq(p-q) = 0$. The peculiar looking form of the conditions follows from the identitity $(3, 6)_{q,p} = (3, 6)_{p+q,-q}$ (also $p = 0$ or $q = 0$ is possble) Note that the tesselation $(3, 6)_{2,2}$ is invariant under reflection and thus non-chiral.

3. The number V of vertices of the triangular itesselation is given by $V = p^2 + q^2 + pq$. The regular tesselation $(p, q) = (2, 2)$ has 12 vertices and is the interesting one in the recent case. It is the smallest regular tesselation. For given (p, q) one can have several non-equivalent pairs (p, q) defining combinatorially non-equivalent tesselations. My interpretation is that they correspond to different conformal equivalence classes for torus: the intuitive expectation is that this should not affect the topology of tesselation nor Hamiltonian cycles. For $(6, 3)_{p,q} = (6, 3)_{2,2}$ with s $(V = 12, F = 24)$ there are 1+6 =7 combinatorially non-equivalent tesselations: one non-chiral and 6 chiral ones.

   Quite generally, the tesselations with V vertices with $V \ mod \ 4 = 0$ (as in the case of $V = 12$) allow one map (chart consisting of faces) with isotropy group of order 2 and 6 maps with isotropy group of order 4. These variants are labelled by an SL(2,Z) matrix $(a, b; 0, c)$ with determinant equal to $V = ac$. For $V = 12$ one has decompositions $12 = 1 \times 12$, $12 = 2 \times 6$, $12 = 3 \times 4$. $-c < b < a - c$ is unique modulo a. In the recent case one as $ac = 12$ allowing $(a, c) \in \{(1, 12), (2, 6), (3, 4)\}$ and pairs obtained by permuting a and c. These matrices need not define combinatorially different tesselations since modular transformations generate equivalent matrices.

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